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Aspects of a representation of quantum theory in terms of classical probability theory by means of integration in Hilbert space

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Abstract. A representation of quantum mechanics in terms of classical probability theory by means of integration in Hilbert space is discussed. This formal hidden-variables representation is analysed in the context of impossibility proofs concerning hidden-variables theories. The structural analogy of this formulation of quantum theory with classical statistical mechanics is used to elucidate the difference between classical mechanics and quantum mechanics.

1. Introduction

Whereas the formalism of quantum mechanics is well accepted in the form proposed by the founders of the theory, the interpretation of this formalism is still a subject of discussion. For a general introduction to the formalism of quantum theory see Jauch (1968) and for information concerning the interpretation of the formalism cf Jammer (1974). Today the Copenhagen interpretation, which is represented in most of the textbooks of quantum theory, has become the orthodox point of view, whereas the statistical interpretation has attracted more and more followers in recent years.

The logical reason for the ability to interpret quantum theory can be traced to the fact that the formalism contains theoretical terms without empirical counterparts. These are the wavefunctions or the vectors of Hilbert space. Any interpretation of quantum theory is devoted to clarifying the meaning of these quantities.

Here we are concerned with the consequences of one aspect in which the familiar interpretations differ from one another. Whereas the Copenhagen interpretation declares that the wavefunction gives the complete description of the state of an individual system, the statistical interpretation insists that the wavefunction refers to a probabilistic situation which empirically can only be realised by an ensemble. The Copenhagen interpretation regards quantum mechanics as an enlargement of the classical mechanics of individual systems, whereas the statistical interpretation—as emphasised by Einstein (1936)—considers it as an enlargement of classical statistical mechanics. As a result, in the frame of the statistical interpretation, hidden variables are possible but are not required. The statistical interpretation admits that the expectation value of quantum mechanics may be expressed as an expectation value in the sense of classical probability theory. The sample space of this probabilistic formulation is built by the hidden variables which are assumed to be governed by a hidden deterministic subdynamics.

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Despite the fact that many famous impossibility proofs exist for hidden-variables theories, we recently have shown (see Bach 1979) that it is possible to express quantum mechanical expectation values by means of integration in Hilbert space. For reasons which will become more obvious in the following sections, we have called this formal hidden-variables theory a probabilistic formulation of quantum theory. The aim of this paper is to discuss this new representation of quantum theory. After introducing the formalism of probabilistic quantum mechanics we investigate the meaning of some impossibility proofs in the context of this representation. On the other hand, as we have particularly derived a representation of quantum theory which structurally is equivalent to classical statistical mechanics, we elucidate the differences between classical mechanics and quantum mechanics. Finally we try to analyse the meaning of probability in quantum theory.

2. Probabilistic quantum mechanics

The central expression in the axioms of quantum theory which connects the empirical quantities with the terms of the formalism is the expectation value. Representing observables by self-adjoint operators A and the state of the system by statistical operators W, the expectation value of the observable A, given the state W, is expressed as

$$E_W(A) = \mathrm{Tr}(WA). \tag{1}$$

The statistical operator W is a positive, self-adjoint operator of the trace class on a complex separable Hilbert space H. The fact that covariance operators of probability measures, defined on a separable Hilbert space H, are positive and self-adjoint operators on H has been the central observation from which the probabilistic representation results. For probability measures in infinite-dimensional spaces, especially real Banach spaces, cf Kuo (1975).

A probability measure μ in H is a measure on the Borel field $\mathscr{B}(H)$ generated by the open subsets of the topological vector space H which satisfies $\mu(H) = 1$. The covariance operator C of a probability measure μ on H is defined by

$$\langle \xi | C | \psi \rangle = \int_H \mathrm{d}\mu(\phi) \langle \xi | \phi \rangle \langle \phi | \psi \rangle, \tag{2}$$

where $\xi, \psi, \phi \in H$.

The fact that the statistical operator W has the defining properties of a covariance operator is used now to define a probability measure μ_W on $(H, \mathscr{B}(H))$ which is characterised by the fact that W is just its covariance operator. We remark that this measure is by no means uniquely determined. By analogy with the procedure in Bach (1979, 1980) this measure allows us to establish the fundamental identity

$$\mathrm{Tr}(WA) = \int_H \mathrm{d}\mu_W(\phi) \langle \phi | A | \phi \rangle \tag{3}$$

which represents the probabilistic formulation of quantum theory. The state space of the system, namely H, constitutes the sample space, and the state and the observable are represented by the measure μ_W and the measurable function $f_A \in L^1(H, \mu_W)$,

$$f_A(\phi) = \langle\phi|A|\phi\rangle, \tag{4}$$

respectively.

Obviously the quantum mechanics of this formulation in terms of classical probability theory is contained in the measurable function $f_A(\phi)$ which expresses the expectation value of the observable A in the unnormalised state $W_\phi = |\phi\rangle\langle\phi|$, $\phi \in H$. In contrast to the conventional expression (1) where all matrix elements of A with respect to a complete orthonormal system are needed, the present representation continuously sums up the (uncountable) set of all diagonal elements of A. In combination with an appropriate measure this is sufficient for a reconstruction of equation (1) as all matrix elements of a self-adjoint operator can be expressed in terms of the diagonal ones by means of the polarisation identity.

We remark that the derivation of formula (3) is confined to bounded operators A. This depends on the fact that, for unbounded operators, the function f_A becomes singular on that subset of H where A is not defined. For the representation of quantum theory, however, formula (3) is quite sufficient as the quantum mechanical lattice of propositions, given by the projections on closed subspaces of H, contains bounded operators only.

3. Connection with hidden-variables theories

Using equation (3) we have derived a formulation of quantum theory which fulfils the criteria of a hidden-variables theory as they are given e.g. by Jammer (1974). Thus we have derived, at least formally, a hidden variables representation of quantum theory. However, the situation is quite different regarding the physical content: the hidden variables are the elements of the state space and the fact that the elements of Hilbert space have no empirical meaning indicates that the theory still remains open to interpretation. Nonetheless, it is interesting to investigate some impossibility proofs in the context of the present representation. As the probabilistic formulation is mathematically equivalent to the conventional one it is obvious that all the impossibility proofs remain valid.

The first and most famous proof for the impossibility of hidden variables has been given by von Neumann (1932). As some assumptions made in this proof have been regarded as physically insufficient, let us remember that Gleason (1957) has proved that each probability measure π on the lattice of quantum mechanical propositions is characterised by a positive, self-adjoint operator W which satisfies $\mathrm{Tr}(W) = 1$ such that

$$\pi(U) = \mathrm{Tr}(P_U W) \tag{5}$$

where P_U denotes the projector on the closed subspace U of H. This theorem can be regarded as a justification of formula (1). By means of equation (1) von Neumann demonstrated that quantum theory does not admit dispersion-free states. Here a dispersion-free state W is defined by the property that

$$\mathrm{Tr}(WA^2) - (\mathrm{Tr}(WA))^2 = 0 \tag{6}$$

holds for all observables A. Inserting for A a one-dimensional projector, von Neumann concluded that (6) cannot hold in general. The fact that there are no dispersion-free states in quantum theory seems to contradict a representation of quantum theory in terms of classical probability theory as every measure space admits a Dirac measure which is dispersion-free.

In the probabilistic representation of quantum theory equation (6) reads

$$\int_H \mathrm{d}\mu_W(\phi) f_{(A^2)}(\phi) - \left(\int_H \mathrm{d}\mu_W(\phi) f_A(\phi)\right)^2 = 0. \tag{7}$$

This differs from the expression

$$\int_H \mathrm{d}\mu_W(\phi)(f_A(\phi))^2 - \left(\int_H \mathrm{d}\mu_W(\phi) f_A(\phi)\right)^2 = 0 \tag{8}$$

as in general

$$f_{(A^2)} \neq (f_A)^2. \tag{9}$$

In contrast to (7), equation (8) is the definition of a dispersion-free random variable f_A on $(H, \mathscr{B}(H))$. From this it becomes obvious that the quantum mechanical meaning of dispersion-free is different from that of classical probability theory. This is due to the fact that the mathematical structure which is used to define probability in quantum theory by means of equation (5) is completely different from the classical lattice of propositions which constitutes the usual σ-algebra of probability theory. Although we give a representation of quantum mechanics in terms of classical probability theory, the concepts of classical probability theory are not appropriate for quantum theory. Moreover, the meaning of dispersion-free in equation (8) refers to all random variables on H whereas the present formalism only admits special bilinear functions, a situation which will be investigated later on. Contrary to this, the measurable function f_A cannot be regarded as an ordinary random variable on H. Equation (9) already shows that f_A does not transform like a classical random variable with respect to operator-valued functions $k(A)$. This means that

$$f_{k(A)} = k(f_A) \tag{10}$$

does not hold.

The fact that, for an arbitrary probabilistic formulation of quantum theory, equation (10) is violated has already been pointed out by Kochen and Specker (1967). They used this fact as an argument for the impossibility of hidden-variables theories. In the present formalism the failure of (10) results from the fact that the empirical variables of $f_A(\phi)$ are the operators on H and not the vectors of H.

The theorem of Bell (1964) states that no local hidden-variables theory for quantum mechanics exists. In the present formalism the condition of locality is defined by

$$f_{AB} = f_A f_B \tag{11}$$

for all observables A, B. Obviously this equation cannot hold in general as the mapping f does not preserve the algebraic structure of the space of observables. This indicates that probabilistic quantum mechanics is a non-local theory and may imply an action-at-a-distance which, however, refers to the Hilbert space and not to the empirical space.

To summarise, we can state that it is not the measure but the measurable function from which the impossibility of a dispersion-free representation results. This agrees with the general impossibility proof of Jauch and Piron (1963) which implies that no deterministic or causal subdynamics exist. As we have explicitly set up a formulation of quantum theory which satisfies the conditions of a hidden-variables theory as it is defined e.g. by Jammer (1974), from an empirical point of view it seems more appropriate to define a hidden-variables theory by the criterion of deterministic subdynamics.

4. The formal analogy between classical statistical mechanics and quantum mechanics

We have derived a representation of quantum mechanics which structurally is quite equivalent to the formulation of classical statistical mechanics. From now on it is necessary to discern between a quantum state in the empirical sense which is given by the statistical operator W and a quantum state in the sense of classical statistical mechanics which is expressed by μ_W. As we stressed when introducing the measures in state space, these are not uniquely determined; they are arbitrary up to the covariance operator. As an example let us consider the pure state $W = |\phi\rangle\langle\phi|$, $|\phi| = 1$. This state can be represented by the Dirac measure

$$\delta_\phi(B) = \begin{cases} 1 & \text{iff} \quad \phi \in B \\ 0 & \text{iff} \quad \phi \notin B \end{cases} \qquad B \in \mathscr{B}(H). \tag{12}$$

On the other hand, the state under consideration can also be represented by a Gaussian measure μ_G on H with zero mean. These measures which represent the same state are quite different as δ_ϕ is concentrated on $\phi \in H$ such that

$$\delta_\phi(\phi) = 1 \tag{13}$$

whereas μ_G is a continuous measure which implies

$$\mu_G(\phi) = 0. \tag{14}$$

In the empirical sense a state in $(H, \mathscr{B}(H))$ can be defined as an element of the quotient space which is obtained from the space of probability measures if equipped with the equivalence relation

$$\mu_{W_1} \sim \mu_{W_2} \quad \text{iff } W_1 = W_2. \tag{15}$$

So, given a statistical operator W, the measure μ_W is nearly arbitrary. Here it is not the elements of the sample space which are unknown; the quantity which really is hidden in quantum theory is the measure in state space. Einstein (cf Einstein *et al* (1935)) has argued that the formalism of quantum theory is not complete. The probabilistic formulation demonstrates the point at which quantum mechanics can be thought of as incomplete.

The indeterminate nature of the measure results from the fact that we describe probabilities in a non-Boolean lattice in terms of classical probability theory. From the empirical point of view, the property that a state μ_W contains much more information than that which can be experimentally verified is connected with the structure of the measurable function f_A. These are special quadratic functionals which only permit us to check the covariance operator. Preparation and measurement, expressed in the structure of f_A, do not allow us to detect the other moments of the measure.

This becomes more obvious by investigation of the localisability of a system in its state space. Denoting by χ_B the characteristic function of the set $B \in \mathscr{B}(H)$, the quantity

$$\mu_W(B) = \int_H \mathrm{d}\mu_W(\phi)\, \chi_B(\phi) \tag{16}$$

is constructed by analogy with the formula of classical statistical mechanics which determines the probability of localising the system in a certain region of phase space. Due to the indeterminate nature of the measures, (16) describes no empirical probability at all as the characteristic function cannot be expressed as a quadratic functional.

For a mixture W, $W^2 < W$, the indeterminate nature of the corresponding measures is related to the fact that a representation of a mixture in terms of pure states need not be unique, such that a convex combination of pure states in quantum mechanics has a meaning which is different from a superposition of states in classical mechanics. The pure states contained in the mixture permit no definite conclusion concerning the question of whether or not a certain subspace of H is occupied by the system. These properties carry over to the associated measures. Representing, e.g., the mixture W by means of a convex combination of Dirac measures, we reveal that this measure (and its support) depends on the special choice of pure states which build up the mixture. However, we need not confine ourselves to convex combinations of Dirac measures. A Gaussian measure with covariance operator W and zero mean has a closed subspace of H as its support but describes the same state.

From these properties we conclude that a state on $(H, \mathscr{B}(H))$ contains no information which permits a localisation of the system in its state space. The probabilistic role of the statistical operator as covariance operator only determines the dispersion of the system; it describes the spreading of the probability but allows no determination of the absolute localisation. All these statements represent the arbitrary manner in which subsets of the state space can be occupied by the system in a definite state and give a quite figurative description of the intrinsic indeterminism of quantum theory.

With respect to quantum dynamics, the indeterminate nature of the measures has immediate consequences. By means of $W(t)$, $t \in [0, T]$, a family of measures $\mu_{W(t)}$ can be constructed which describes a stochastic process with state space $H^{[0,T]}$. As the measures can be arbitrarily chosen at each instant of time, this stochastic process is not uniquely defined. Contrary to the situation in classical mechanics no specific stochastic process for quantum dynamics exists. This is the reason why the evolution in Hilbert space, which is described by the solution of the Schrödinger equation, can by no means be assumed to indicate a deterministic quantum mechanical evolution. There is only one possibility of describing the dynamics of a pure state, namely that one which corresponds to a family of Dirac measures.

5. Conclusion

In this paper we have presented a formulation of quantum theory which differs essentially from the conventional one. Due to the unfamiliar integration techniques in infinite-dimensional spaces, the advantage of the present formalism is not clear at first sight, so that we feel it necessary to discuss the consequences of this fact with respect to the methodological and substantial background.

From the methodological point of view we have shown that quantum theory fits into the scheme of classical probability theory such that all techniques of this discipline become available. As it is always possible to choose a Gaussian measure for the description of the state, the well known theory of Gaussian measures in infinite-dimensional space can be successfully applied. These properties can be regarded as a basis for an analytical application of the present formalism.

The far reaching indeterminism involved in this formulation of quantum theory seems to indicate that this representation is somewhat artificial. On the one hand, this property may be helpful by the flexibility it implies, particularly in connection with the mathematical elaboration. On the other hand, the indeterminism clearly shows the difference between classical and quantum physics. The representation of quantum theory by means of classical probability theory enables us to perform a detailed comparison of these theories which cannot be realised by means of the conventional representation which uses the theory of linear transformations in Hilbert space. We have discussed this aspect in the last section and we are left to consider the consequences with respect to the interpretation of quantum theory. Our representation demonstrates quite clearly that the substantial difficulties of the interpretation of quantum mechanics relate to the meaning of probability. Both the Copenhagen interpretation and the statistical interpretation have aspects which are well confirmed by the probabilistic representation. This formulation shows how far quantum mechanics can be regarded as an enlargement of classical mechanics and where it is incomplete. On the other hand, this representation verifies that there is no deterministic subdynamics; therefore quantum mechanics is empirically complete. In the present context, this interplay between completeness and indeterminism is expressed by the properties of the measurable functions and the measures respectively.

The failure to construct deterministic subdynamics, which has been proved by Jauch and Piron (1963), shows that probability in quantum theory cannot have the same meaning as in classical statistical mechanics. In particular, it cannot be thought of as expressing lack of knowledge. Probability in quantum theory is an intrinsic property of the system which surpasses all concepts of the subjective or objective indeterminateness of classical probability. It is an element which belongs to the fundamental structure of a physical system.

Here we have explicitly shown that the measures or even the properties are nearly undetermined. This is due to the fact that a subset of the set of observables can be equally well regarded as statistical operators and that all statistical operators can, in principle, represent an observable. In contrast to classical mechanics, where a measure in state space is an element which can be verified empirically, a state in quantum theory, if represented in terms of classical probability theory, is fictive besides the fact that the covariance operator describes the spreading of probability in Hilbert space. From this we conclude that *classical probability in quantum theory is a theoretical term which has no empirical meaning.*

Quantum mechanics shows two stages of indeterminacy. One is expressed by the fact that it is possible at all to derive a probabilistic representation: it is classical probability. The other one—the stage of subdynamics—is expressed by the indeterminate nature of the measures. This is where quantum probability surpasses classical probability and from which we learn that classical probability need not refer to an empirical situation. The indeterminism of quantum mechanics, however, belongs to the empirical part of the system as it can be verified experimentally.

From the empirical point of view, probability in quantum mechanics is determined as everywhere else in physics, namely by measurement of an ensemble. From this point of view there is no difference between the Copenhagen interpretation and the statistical interpretation; both of them refer to the description of the state of a quantum system which is expressed in terms of classical probability—the only theory of probability at the time when these interpretations were proposed—by means of a measure in Hilbert space. As these probability measures are theoretical terms which have no empirical meaning, the major difference between the above mentioned interpretations, namely whether the state refers to an individual system or an ensemble, becomes irrelevant as these terms refer to empirical quantities.

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